ISRAEL JOURNAL OF MATHEMATICS, Vol. 71, No. 3, 1990

ON TRIANGULAR SUBALGEBRAS OF GROUPOID C^*-ALGEBRAS

BY

PAUL S. MUHLY[a], † AND BARUCH SOLEL[b], ‡

[a]*Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA; and*
[b]*Department of Mathematics, University of North Carolina at Charlotte, Charlotte, ND 28223, USA*

ABSTRACT

Let $\mathfrak{B}$ be an AF C^*-algebra with Stratila–Voiculescu masa $\mathfrak{D}$ and let $\mathfrak{A}$ be a maximal triangular subalgebra of $\mathfrak{B}$ with diagonal $\mathfrak{D}$. Peters, Poon and Wagner showed that $\mathfrak{A}$ need not be a C^*-subdiagonal subalgebra of $\mathfrak{B}$ in the sense of Kawamura and Tomiyama. We investigate and explain this phenomena here from the perspective of groupoid C^*-algebras by representing $\mathfrak{A}$ as the "incidence algebra" associated with a topological partial order. A number of examples are given showing what can keep a maximal triangular algebra from being C^*-subdiagonal.

§1. Introduction

Our objective in this note is to combine the analysis of [MS] with [PPW] and [T] to help clarify the structure of maximal triangular subalgebras of AF C^*-algebras and to resolve an issue that arose in the writing of [MS].

We follow the notation and terminology of [R] and [MS] with regard to groupoids. Suppose that G is a 2nd countable, locally compact, r-discrete, amenable, principal groupoid admitting a Haar system and a cover by compact open G-sets, suppose σ is a two cocycle on G with values in $\mathbf{T}$ and let $C^*(G,\sigma)$ be the associated C^*-algebra. In [MS], we show that given a norm closed subalgebra $\mathfrak{A}$ of $C^*(G)$ such that $\mathfrak{A} \cap \mathfrak{A}^* = C^*(G^0)$ (such an algebra is called *triangular*), then there is an open subset $P \subseteq G$ such that $P \cdot P \subseteq P$ and $P \cap P^{-1} = G^0$ and such that $\mathfrak{A}$ is the closure in $C^*(G)$ of the set of all functions in $C_c(G)$ that are supported in P. We write $\mathfrak{A} = \mathfrak{A}(P)$. The correspondence between triangular algebras and open sets P with the indicated properties is one to one, so the notation is justified. We note that our terminology differs slightly from that of Peters, Poon and

†Supported by a grant from the National Science Foundation.

‡*Current address:* Department of Mathematics, Technion, Haifa, 32000 Israel.

Received November 15, 1989

Wagner. Their triangular algebras need not be closed. This difference will be of no consequence here.

Groupoids of the kind we are considering may be viewed as equivalence relations on G^0, i.e., certain subsets of $G^0 \times G^0$, but with topologies that may be finer than the product topology. The sets P, then, are (the graphs of) partial orders on G^0 and the algebras $\mathfrak{A}(P)$ may be viewed as one possible generalization of incidence algebras.

A triangular algebra $\mathfrak{A}(P)$ is called *maximal* triangular if it is not contained in any larger triangular subalgebra. In terms of the set P, this means that P is not contained in any larger partial order on G^0. If G^0 were finite, then this would be the case if and only if $P \cup P^{-1} = G$, i.e., if and only if P totally orders each equivalence class determined by G. When [MS] was written, we did not know if a maximal partial order has this property in general. This was important for [MS] because an algebra $\mathfrak{A}(P)$, with P satisfying $P \cup P^{-1} = G$, has additional structures that makes it tractable for analysis. Indeed, such algebras turn out to be what are known as maximal C^*-subdiagonal algebras [KT] and share many structural properties with the full algebra of upper triangular matrices. One of the consequences of [PPW] is that maximal triangular algebras $\mathfrak{A}(P)$ *do* exist for which P fails to satisfy $P \cup P^{-1} = G$. Moreover, examples may be found in the context of AF C^*-algebras. Recall [R, II.1.15] that every AF C^*-algebra may be realized in terms of an AF groupoid (and in that case $\sigma \equiv 1$). Analysis of their examples as well as the examples in [T] show that the sets P constructed are open, but not closed. However, if P is open and $P \cup P^{-1} = G$, then P is closed because $P = G^0 \cup G \backslash P^{-1}$. This and Proposition 1.1 lead one to speculate that if P corresponds to a maximal triangular algebra and if P is closed, then $P \cup P^{-1} = G$. The main example of this note, Example 3.7, shows that this is not the case. Its construction is rather complicated and its existence is rather surprising. The latter calls for an effective criterion for deciding when an open partial order in an AF groupoid is closed. This is the goal of Theorem 1.3, below, but before giving it we present Proposition 1.1 giving a usable sufficient condition implying $P \cup P^{-1} = G$ when $\mathfrak{A}(P)$ is maximal. For it, recall that $\mathfrak{A}(P)$ is called a *nest algebra* if there is a totally ordered family, $\{p_\alpha\}_{\alpha \in A}$, of projections in the multiplier algebra of $C^*(G^0)$ such that

$$\mathfrak{A}(P) = \{a \in C^*(G,\sigma) | (1 - p_\alpha) a p_\alpha = 0 \text{ for all } \alpha \in A\}.$$

PROPOSITION 1.1. *If $\mathfrak{A}(P)$ is a nest subalgebra of $C^*(G,\sigma)$ and if P is closed, then $P \cup P^{-1} = G$. That is, $\mathfrak{A}(P)$ is a maximal C^*-subdiagonal subalgebra of $C^*(G,\sigma)$.*

PROOF. Let $\{p_\alpha\}_{\alpha\in A}$ be the totally ordered family of projections in the multiplier algebra of $C^*(G^0)$ that determines $\mathfrak{A}(P)$ and write $p_\alpha = 1_{E_\alpha}$, where E_α is a closed and open subset of G^0. Since $\mathfrak{A}(P) = \{a \in C^*(G,\sigma) | (1-p_\alpha)ap_\alpha = 0, \alpha \in A\}$, $P = G\backslash Q$, where

$$Q = \left[\bigcup_{\alpha\in A} (E_\alpha^c \times E_\alpha)\right] \cap G$$

and where E_α^c denotes the complement of E_α in G^0. The sets $E_\alpha^c \times E_\alpha$ are open in $G^0 \times G^0$ and so Q is an open subset of G which, recall, has a topology generally finer than the relative topology on G. Further, since

$$Q^{-1} = \left[\bigcup_{\alpha\in A} (E_\alpha \times E_\alpha^c)\right] \cap G, \quad Q \cap Q^{-1} = \varnothing$$

because the E_α's are ordered by inclusion. Since Q and Q^{-1} are open, the fact that they are disjoint implies that $\bar{Q} \cap Q^{-1} = Q \cap \overline{Q^{-1}} = \varnothing$. Since, however, P is closed by hypothesis, $\bar{Q}$ must be open. Hence, $\bar{Q} \cap \overline{Q^{-1}} = \varnothing$ and so $P \cup P^{-1} = (G\backslash\bar{Q}) \cup (G\backslash\overline{Q^{-1}}) = G\backslash(\bar{Q} \cap \overline{Q^{-1}}) = G$. ∎

REMARK 1.2. Proposition 1.1 is reminiscent of Theorem 3.1.1 and Corollary 3.1.2 in [A], but the two proofs are entirely different.

Recall that an AF C^*-algebra A can be written as an inductive limit $A = \varinjlim(A_n, i_n)$ in many different ways. Here A_n denotes a finite dimensional C^*-algebra and i_n is a $*$-injection mapping A_n into A_{n+1}. To express A as $C^*(G)$ for an AF groupoid is to make explicit and to keep track of some additional information. Namely, this amounts to picking masas $D_n \subseteq A_n$ such that $i_n(D_n) \subseteq D_{n+1}$, so that if the normalizer of D_n in A_n is denoted by $\mathfrak{N}(D_n)$, then $i_n(\mathfrak{N}(D_n)) \subseteq \mathfrak{N}(D_{n+1})$. (Recall that $\mathfrak{N}(D_n) = \{v \in A_n | v$ is a partial isometry, and vav^*, $v^*av \in D_n$ for all $a \in D_n\}$.) Of course, when the A_n's are viewed as subalgebras of $C^*(G)$, $D_n = A_n \cap C^*(G^0)$ and $C^*(G^0)$ may be viewed as $D = \varinjlim(D_n, i_n)$. We will frequently write $D = C^*(G^0)$. A key result of Power [P, Lemma 1.3] asserts that if $M \subseteq A$ is a norm-closed bimodule over D, i.e., if DM and $MD \subseteq M$, then there is a (unique) D_n-bimodule M_n in A_n such that $M = \varinjlim(M_n, i_n)$. Thus, as one says, bimodules over D are *inductive*. Hence, given a triangular subalgebra $T = \mathfrak{U}(P)$ in $C^*(G)$, we may write $T = \varinjlim(T_n, i_n)$ where the T_n are triangular subalgebras of A_n, i.e., $T_n \cap T_n^* = D_n$. We call $\{T_n\}$ the *generating sequence* of T relative to $\{(A_n, i_n)\}$. Following [PPW], we say that $T = \mathfrak{A}(P)$ is *strongly maximal* if it is possible to realize A as $\varinjlim(A_n, i_n)$, with $D = \varinjlim(D_n, i_n)$ in such a way that $T_n + T_n^* = A_n$. Finally, we call a generating sequence $\{T_n\}$ for a triangular algebra

T closed if for every n, every matrix unit e in $A_n \backslash T_n + T_n^*$, and every $d \in D_{n+1}$, $di_n(e)$ lies in $A_{n+1} \backslash (T_{n+1} + T_{n+1}^*)$. The following is the principal general result of this paper. The first part was proved in [T] (cf. Proposition 3.2) by somewhat different means.

THEOREM 1.3. *Suppose that $T = \mathfrak{A}(P)$ is a maximal triangular subalgebra in the AF C^*-algebra $A = C^*(G)$ with respect to $D = C^*(G^0)$.*

1. *The following three assertions are equivalent.*

(1.1) *T is strongly maximal.*

(1.2) $P \cup P^{-1} = G$.

(1.3) *$T + T^*$ is dense in $C^*(G)$.*

2. *The following assertions are equivalent.*

(2.1) *T has a closed generating sequence.*

(2.2) *P is closed.*

(2.3) *There is a norm-closed D-bimodule B such that $[T + T^* + B]^{cl} = C^*(G)$ while $B \cap [T + T^*]^{cl} = \{0\}$.*

The proof is given in Section 2.

With regard to 1. in the theorem, we find that the strongly maximal triangular subalgebras of $C^*(G)$ are precisely the maximal C^*-subdiagonal subalgebras of $C^*(G)$ when $C^*(G)$ is AF. This fact was proved by Ventura in [V] using different means. It follows by definition of "closed generating sequences" that a strongly maximal triangular algebra in an AF C^* algebra has a closed generating sequence. We also see this from Theorem 1.3 since, as we noted earlier, if $G = P \cup P^{-1}$ while $P \cap P^{-1} = G^0$, where P is open, then $P = G^0 \cup G \backslash P^{-1}$ is closed.

To find our example, Example 3.7, of a maximal triangular algebra $T = \mathfrak{A}(P)$, where P is closed but does not satisfy $P \cup P^{-1} = G$, we use Theorem 1.3 and show that a maximal triangular algebra T exists in an AF C^*-algebra that has a closed generating sequence, but is not strongly maximal. We also elaborate on examples in [PPW] and [T] showing that maximal triangular algebras exist for which there are no closed generating sequences, Examples 3.1–3.3. In particular, we show in Example 3.2 that there is a maximal triangular *nest* subalgebra of an AF algebra that has no closed generating sequence.

§2. Proof of Theorem 1.3

Throughout, we suppose that our AF C^*-algebra A is realized as $C^*(G)$ for a fixed AF groupoid G with D represented by $C^*(G^0) = C_0(G^0)$. We will write

$G = \bigcup G_n$ where $G_n \subseteq G_{n+1}$ and each G_n is a closed and open elementary subgroupoid of G. Recall [R, III.1.1] that this means that G_n can be written as the countable disjoint union $\bigcup X_i \times \{1,2,\dots,k_i\}^2$ where X_i is a zero dimensional locally compact Hausdorff space and $\{1,2,\dots,k_i\}^2$ denotes the trivial (or transitive) groupoid on the finite set $\{1,2,\dots,k_i\}$. The unit space G^0 of G, then, is $\bigcup X_i$ and $C^*(G_n) \cong \sum M(k_i) \otimes C_0(X_i)$, where $M(k_i)$ denotes the algebra of $k_i \times k_i$ complex matrices. The algebra $C^*(G)$, then, is the nested union of these special C^*-algebras.

Proposition 2.1. *Suppose that P is an open subset of G satisfying*

(2.1) $P \cdot P \subseteq P$,

(2.2) $P \cap P^{-1} = G^0$,

(2.3) *P is closed, and*

(2.4) *P is maximal with respect to conditions* (2.1) *and* (2.2).

Then $\mathfrak{A}(P)$ is a maximal triangular AF C^*-algebra with a closed generating sequence. Moreover, if $P \cup P^{-1} = G$, then $\mathfrak{A}(P)$ is strongly maximal.

Proof. Since every bimodule over D is of the form $\mathfrak{A}(Q)$ for some open set $Q \subseteq G$ (where $\mathfrak{A}(Q) := \{f \in C^*(G) | \operatorname{supp} f \subseteq Q\}$), we see that $\mathfrak{A}(P)$ is a maximal triangular subalgebra of $C^*(G)$ if P satisfies (2.1), (2.2) and (2.4). We need to show that if P is closed, then $\mathfrak{A}(P)$ admits a closed generating sequence.

To this end, write $P_n = G_n \cap P$, so that $P_n^{-1} = G_n \cap P^{-1}$, and note that P_n and P_n^{-1} are closed and open subsets of G_n. Suppose first that G_n is an elementary groupoid of type k, so that $G_n = X \times \{1,2,\dots,k\}^2$. Write $\Omega_k = 2^{\{1,\dots,k\}^2}$, the set of all subsets of $\{1,2,\dots,k\}^2$, and let

$$\omega(x) = \{(i,j) \in \{1,2,\dots,k\}^2 | (x,(i,j)) \in P_n\},$$

$x \in X$. Thus $\omega(\cdot)$ is a map from X onto Ω_k. For fixed $\omega \in \Omega_k$, let $X_\omega = \{x \in X | \omega(x) = \omega\}$. Then

$$X_\omega = \left(\bigcap_{(i,j)\in\omega} \{x \in X | (i,j) \in \omega(x)\} \right) \cap \left(\bigcap_{(i,j)\notin\omega} \{x \in X | (i,j) \notin \omega(x)\} \right).$$

Since, for a fixed (i,j), the set $\{x \in X | (i,j) \in \omega(x)\} = \{x \in X | (x,(i,j)) \in P_n\}$ is closed and open (because the map $x \to (x,(i,j))$ is continuous and P_n is closed and open), we see that X_ω is closed and open. Clearly, X is the disjoint union of the X_ω's. Hence G_n is the disjoint union $\bigcup_{\omega\in\Omega_k} (X_\omega \times \{1,2,\dots,k\}^2)$ while $P_n = \bigcup_{\omega\in\Omega_k} (X_\omega \times \omega)$. All this was done for an elementary groupoid of type k. In general, G_n is the countable disjoint union $G_n = \bigcup_i X_i \times \{1,\dots,k_i\}^2$ and so we may write

$$G_n = \bigcup_{\omega_i \in \Omega_{k_i}} (X_{i,\omega} \times \{1,2,\dots,k_i\}^2) \quad \text{and} \quad P_n = \bigcup_{\omega_i \in \Omega_{k_i}} (X_{i,\omega} \times \omega),$$

where the $X_{i,\omega}$ are closed and open.

For each $(j,m) \in \{1,\dots,k_i\}^2$ and every $\omega \in \Omega_{k_i}$, the set $s = X_{i\omega} \times \{(j,m)\}$ is a closed and open G-set with $\chi_s \in C^*(G_n) \subseteq C^*(G)$. Let C_n be the C^*-subalgebra of $C^*(G)$ that is generated by $\{\chi_s \mid s = X_{i\omega} \times \{(j,m)\},\ (j,m) \in \{1,2,\dots,k_i\}^2,\ \omega \in \Omega_{k_i}\}$. Then C_n is a C^*-subalgebra of $C^*(G_n)$ isomorphic to $A_n := \sum_i \oplus M(k_i)$. This isomorphism sends χ_s, $s = X_{i\omega} \times \{(j,m)\}$, into the (j,m)-matrix unit in $M(k_i)$. The algebra, $C_n(P)$, generated by

$$\{\chi_s \mid s = X_{i\omega} \times \{(j,m)\},\quad (j,m) \in \omega\},$$

is sent onto a triangular subalgebra T_n of A_n. Observe that if $P \cup P^{-1} = G$, then $P_n \cup P_n^{-1} = G_n$ and T_n is a maximal triangular subalgebra of A_n.

Observe, too, that A_n is finite dimensional if and only if the number of X_i's is finite. In general, the A_n are themselves AF, but in a trivial sort of way, and it would belabor the notation and discussion to make allowance for this possibility. Consequently, we will proceed with our discussion, treating the A_n as if they were finite dimensional and leaving it to the reader to make the necessary changes when they aren't.

The containment $C_n \subseteq C_{n+1}$ induces an inclusion map in $A_n \to A_{n+1}$ that sends T_n into T_{n+1}. Also, every matrix unit in A_n, which may be viewed as a χ_s with s as above, is a sum of matrix units in A_{n+1}. Hence $\mathfrak{N}(D_n) \subseteq \mathfrak{N}(D_{n+1})$. Since it is clear that $\bigcup C_n$ is dense in $C^*(G)$, we see that $C^*(G) \cong \varinjlim (A_n, i_n)$.

To see that $\mathfrak{A}(P) \cong \varinjlim T_n$, we show that $\mathfrak{A}(P) = \overline{\bigcup_n C_n(P)}$. Let $\mathfrak{G}$ denote the ample semigroup of compact open G-sets [R, I.2.10] and suppose that an $s \in \mathfrak{G}$ is contained in P. Then for every $x \in r(s)$, $(x, s(x))$ is contained in P_n for some n depending on x. Since G_n is open, there is a neighborhood U_x of x such that $(y, s(y)) \in P_n$ for all $y \in U_x$. Since $r(s)$ is compact, $r(s)$ is covered by finitely many of the U_x and so there is an n such that $s \subseteq P_n$. Hence, we may find a $t \in \mathfrak{G}$ such that $s \subseteq t$ and $\chi_t \in C_n(P)$. It follows from the proof of [R, III.1.15] that every projection in $C^*(G^0)$ lies in $\bigcup_k C_k(P)$. Hence $\chi_s \in \bigcup_k C_k(P)$, and this shows that $\mathfrak{G}(P) = \overline{\bigcup C_n(P)}$.

It remains to show that $\{T_n\}_{n=1}^{\infty}$ is a closed generating sequence. Suppose e is a matrix unit in A_n that is not in $T_n + T_n^*$. Then e may be viewed as an element χ_s in C_n where $s = X_{i\omega} \times \{(j,m)\}$ for some i, ω, j and m. (Note: The closedness of P was used to conclude that s is closed and open.) Since $e \notin T_n + T_n^*$, by hypothesis, $(j,m) \notin \omega$ and $(m,j) \notin \omega$. Thus $s \cap (P \cup P^{-1}) = \emptyset$. (Remember: $s \subseteq G_n = \bigcup_{\omega_i \in \Omega_{k_i}} X_{i\omega} \times \{1,2,\dots,k_i\}^2$ and

$$P_n \cup P_n^{-1} = \left(\bigcup_{\omega_i \in \Omega_{k_i}} (X_{i,\omega} \times \omega) \right) \cup \left(\bigcup_{\omega_i \in \Omega_{k_i}} (X_{i,\omega} \times \omega^{-1}) \right),$$

so if $(j,m) \notin \omega$ and $(m,j) \notin \omega$, then s is disjoint from $P \cup P^{-1}$.) Hence for every $d \in D_{n+1}$, $di_n(e)$ does not lie in $T_{n+1} + T^*_{n+1}$, and the proof is complete. ■

LEMMA 2.2. *Suppose T is a maximal triangular subalgebra of the AF C^*-algebra $A = \varinjlim(A_n, i_n)$ and suppose T has a closed generating sequence $\{T_n\}_{n=1}^{\infty}$. Then $\{T_n\}_{n=1}^{\infty}$ is maximal in the sense that if $T_n \subseteq S_n \subseteq A_n$ for all n and $T = \varinjlim(S_n, i_n)$, then $T_n = S_n$ for all n.*

PROOF. Suppose for some n_0, $S_{n_0} \neq T_{n_0}$. Then there is a matrix unit $e \in S_{n_0} \setminus (T_{n_0} + T^*_{n_0})$. By hypothesis, $i_{n_0}(e) \in S_{n_0+1} \setminus (T_{n_0+1} + T^*_{n_0+1})$. Since $i_{n_0}(e)$ is a sum of matrix units in A_{n_0+1}, $i_{n_0+1}(i_{n_0}(e)) \in S_{n_0+2} \setminus (T_{n_0+2} + T^*_{n_0+2})$. Indeed, for every $j > 0$,

$$i_{n_0+j} \circ i_{n_0+j-1} \circ \cdots \circ i_{n_0}(e) \in S_{n_0+j} \setminus (T_{n_0+j} + T^*_{n_0+j}).$$

Consequently, $\varinjlim(S_n, i_n) \neq \varinjlim(T_n, i_n) = T$. This contradiction completes the proof. ■

PROPOSITION 2.3. *Let $A = \varinjlim(A_n, i_n)$ be an AF C^*-algebra realized as $C^*(G)$ for some AF groupoid G and let $T = \varinjlim(T_n, i_n)$ be a maximal triangular subalgebra of A (with respect to $D = C^*(G^0)$).*

(1) There is an open subset $P \subseteq G$ satisfying $G^0 = P \cap P^{-1}$, $P \cdot P \subseteq P$ and P is maximal with respect to these properties such that $T = \mathfrak{A}(P)$.

(2) If T is strongly maximal, then $P \cup P^{-1} = G$.

(3) If $\{T_n\}_{n=1}^{\infty}$ is a closed generating sequence, then P is closed.

PROOF. (1) It follows from Theorems 3.10 and 4.1 of [MS] that $T = \mathfrak{A}(P)$ for an open subset $P \subseteq G$ satisfying $P \cdot P \subseteq P$ and $P \cap P^{-1} = G^0$. If $Q \supseteq P$ is an open subset of G such that $Q \cap Q^{-1} = G^0$ and $Q \cdot Q \subseteq Q$, then $\mathfrak{A}(Q)$ is a triangular subalgebra of $C^*(G)$ that contains $T = \mathfrak{A}(P)$. Since T is assumed maximal, $\mathfrak{A}(Q) = T = \mathfrak{A}(P)$ and $P = Q$.

(2) If T is strongly maximal, then $C^*(G) = \overline{\bigcup A_n}$, with $T = \overline{\bigcup T_n}$ and $T_n + T_n^* = A_n$. Consequently, $A_n \subseteq T + T^*$ and $C^*(G) = \overline{T + T^*}$. By Theorem 4.2 of [MS], $P \cup P^{-1} = G$ (where $T = \mathfrak{A}(P)$).

(3) Suppose that $\{T_n\}_{n=1}^{\infty}$ is a closed generating sequence. We recall how G and A are related as in the proof [R, III.1.15]. The normalizer $\mathfrak{N}_n$ of D_n $(= T_n \cap T_n^*)$ in A_n is contained in $\mathfrak{N}_{n+1}$. Also $C^*(G^0) = \overline{\bigcup D_n}$ and G^0 is a totally disconnected locally compact Hausdorff space. We write $\mathfrak{G}_n$ for the ample inverse semigroup of D_n. This is the inverse semigroup of maps of D_n induced by $\mathfrak{N}_n$. We view $\mathfrak{N}_n$ as

acting on D_n or on $C^*(G^0)$, and write $\mathfrak{G} = \bigcup \mathfrak{G}_n$. The principal groupoid of the orbit equivalence relation corresponding to $\mathfrak{G}_n$ is denoted G_n. Each G_n is an open and closed subgroupoid of G with $G_n^0 = G^0$ and we have $G = \bigcup G_n$.

Since $T_n \supseteq D_n$, T_n is generated as a D_n-bimodule by the set of elements of $\mathfrak{N}_n$ that it contains. We call this set $\mathfrak{N}_n^+$ and let $\mathfrak{G}_n^+$ be the corresponding subset of $\mathfrak{G}_n$. Evidently, $\mathfrak{G}_n^+ \subseteq \mathfrak{G}_{n+1}^+$ and if we let P_n be the union of the graphs of the elements in $\mathfrak{G}_n^+$, we see that $P_n \subseteq G_n$ and $P_n \subseteq P_{n+1}$. We write $C^*(\mathfrak{G}_n)$ for the finite dimensional C*-algebra generated by $\{\chi_s\}_{s \in \mathfrak{G}_n}$. This algebra is isomorphic to A_n and T_n is isomorphic to the algebra generated by $\{\chi_s \mid s \in \mathfrak{G}_n^+\}$.

We write $P = \cup P_n$, an open subset of G, and we identify T with $\mathfrak{G}(P)$. We wish to show P is closed. First, we show that for every n,

$$G_n \setminus (P_n \cup P_n^{-1}) \subseteq G_{n+1} \setminus (P_{n+1} \cup P_{n+1}^{-1}). \tag{2.1}$$

Suppose this is not the case. Then there is an $s \in \mathfrak{G}_n$ with graph disjoint from $P_n \cup P_n^{-1}$, but not from P_{n+1}. Write $s_0 = s \cap P_{n+1}$. Then $\chi_{s_0} = d\chi_s$ for some d in D_{n+1} and it is in $A_n \setminus T_n + T_n^*$ as well as in T_{n+1}. This contradicts the assumption that $\{T_n\}_{n=1}^{\infty}$ is a closed generating sequence. Consequently (2.1) holds. Now each of the sets $G_n \setminus (P_n \cup P_n^{-1})$ is open and so

$$G \setminus (P \cup P^{-1}) = \bigcup_n \, (G_n \setminus (P_n \cup P_n^{-1}))$$

is also open. But then $P \cup P^{-1}$ is closed and so, too, is $P = G^0 \cup (P \setminus G^0) = G^0 \cup ((P \cup P^{-1}) \setminus P^{-1})$ since G^0 is closed and P^{-1} is open. ■

Proof of Theorem 1.3. (1) The only thing left to show is that $P \cup P^{-1} = G$ if and only if $\overline{T + T^*} = C^*(G)$. This, however, is immediate from Theorem 4.2 of [MS] and its proof.

(2) Suppose P is closed. Then $Q := G \setminus (P \cup P^{-1})$ is open and $Q \cup P \cup P^{-1} = G$. Consequently, $\overline{T + T^* + \mathfrak{A}(Q)} = C^*(G)$. Suppose, on the other hand, that $\overline{T + T^* + B} = C^*(G)$ for some bimodule B with $B \cap \overline{(T + T^*)} = \{0\}$. Then by Theorem 3.10 of [MS], $B = \mathfrak{A}(Q)$ for some open subset $Q \subseteq G$. The equation $B \cap \overline{T + T^*} = \{0\}$ implies that $Q \cap (P \cup P^{-1}) = \varnothing$ while the equation $\overline{B + T + T^*} = C^*(G)$ implies that $Q \cup P \cup P^{-1} = G$. Thus $P \cup P^{-1}$ is closed and $P = G^0 \cup (P \setminus G^0) = G^0 \cup ((P \cup P^{-1}) \setminus P^{-1})$ is closed also. ■

§3. Examples

In this section we supply examples of maximal triangular subalgebras of AF algebras that illustrate the trichotomy that we have been exposing.

EXAMPLE 3.1. There is a maximal triangular algebra in an AF C^*-algebra that does not have a closed generating sequence.

This example is a type discovered by Thelwall in [T]. Let X be a zero dimensional compact Hausdorff space, so that $C(X)$ is AF, and consider $A = M_2(C(X))$, the 2×2 matrices over $C(X)$. Let U and V be two disjoint open subsets of X such that $U \cup V$ is dense in X but every $x \in X \setminus U \cup V$ is an accumulation point of both U and V. Our algebra T is realized as $\{(a_{ij})_{i,j=1}^2 \mid a_{11}, a_{22} \in C(X),\ a_{12} \in C_0(U) \text{ and } a_{21} \in C_0(V)\}$. To visualize the corresponding P, view the equivalence relation G giving $M_2(C(X))$ as the subset of the Cartesian square, $(X \dot{\cup} X)^2$, of the union of two disjoint copies of X consisting of the diagonal and the "diagonal pieces" of the northeast and southwest corners of $(X \dot{\cup} X)^2$ (Fig. 1a). Then P is the diagonal of $(X \dot{\cup} X)^2$ together with the "northeast diagonal" determined by U and the "southeast diagonal" determined by V (Fig. 1b). The fact that U and V are open implies that P is open. Also, since $U \cup V$ is dense in X, $P \cup P^{-1}$ is dense in G. However, since every point $x \in X \setminus U \cup V$ is an accumulation point for both U and V, it is easy to see that P is not contained in any larger open, transitive, reflexive and antisymmetric subset of G. Thus T is maximal triangular, but does not have a closed generating sequence by Theorem 1.3.

EXAMPLE 3.2. There is a maximal triangular nest subalgebra of an AF C^*-algebra that does not have a closed generating sequence.

We use the above example and construct the sets U and V judiciously. Let X be the Cantor set in $[0,1]$. Let $E_0 = X \cap [0,1/2]$, $E_1 = X \cap (1/2,3/4)$, $E_2 = X \cap (3/4,7/8), \ldots$. Each E_i is closed and open in X. Set $U = \bigcup_{i\,\text{even}} E_i$ and set $V = \bigcup_{i\,\text{odd}} E_i$. Then U and V are disjoint open sets, $X \setminus (U \cup V) = \{1\}$ and 1 is an ac-

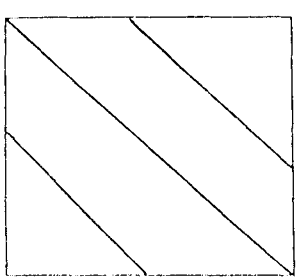

Fig. 1a.

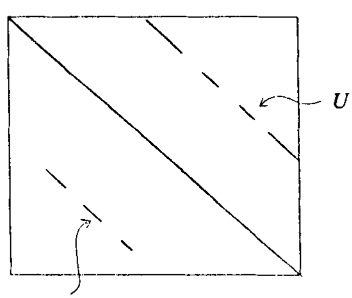

Fig. 1b.

cumulation point of both U and V. Hence the algebra T associated with U and V in the preceding example is maximal triangular, but doesn't have a closed generating sequence. Now let $\{e_{ij}\}_{i,j=1}^{2}$ denote the matrix units of $M(2)$ and regard them as constant elements of $M_2(C(X))$. Define

$$p_0 = e_{11}\chi_{E_0}, \quad p_2 = e_{11}\chi_{E_0} + e_{22}\chi_{E_1}, \quad p_3 = e_1\chi_{E_0} + e_{22}\chi_{E_1} + e_{11}\chi_{E_2},$$

$$\cdots p_{n+1} = p_n + \begin{cases} e_{11}\chi_{E_n}, & n \text{ even}, \\ e_{22}\chi_{E_n}, & n \text{ odd}. \end{cases}$$

Then $\{p_n\}_{n=0}^{\infty}$ is a nest in $M_2(C(X))$ and it is easy to see that $T = \mathrm{Alg}(\{p_n\}_{n=0}^{\infty})$. ∎

The first example of a maximal triangular subalgebra of an AF C^*-algebra that is not maximal triangular is due to Peters, Poon and Wagner [PPW, Example 3.25]. We show that their example does not have a closed generating sequence. It was this example that provided much of the impetus for the present study.

Example 3.3. We follow the notation of Example 3.25 in [PPW]. There, the authors construct a maximal triangular subalgebra $\mathfrak{J}$ of a unital AF-algebra $\mathfrak{A}$ such that $e_{ij}^{(k)} \notin \overline{\mathfrak{J} + \mathfrak{J}^*}$. There is a closed and open G-set $\tau \subseteq G$, where G is an AF groupoid associated with $\mathfrak{A}$, such that $e_{12}^{(1)} = \chi_\tau \in C^*(G)$. Suppose $\mathfrak{J} = \mathfrak{A}(P)$ with P closed. Then $\tau_1 := \tau \cap P$, $\tau_2 := \tau \cap P^{-1}$ and $\tau_3 := \tau \cap (G \setminus (P \cup P^{-1}))$ are all closed and open G-sets. Write $f_i = r(\tau_i)$, $i = 1,2,3$. Then each f_i is a closed and open subset of G^0, the f_i are disjoint and $r(\tau) = f_1 \cup f_2 \cup f_3$. Write F_i for the projection in $C^*(G^0)$ associated with f_i, $i = 1,2,3$. Then $F_1 + F_2 + F_3 = e_{11}^{(1)}$. For every $k \geq 1$, we may write $e_{12}^{(k)} = \chi_{t_k}$, $t_k \subseteq G$. Write $g_k := r(t_k) \subseteq G^0$, so $e_{11}^{(k)} = \chi_{g_k}$. Then $\{g_k\}$ is a decreasing sequence of compact open subsets of G^0 whose intersection, $g = \cap g_k$, is a set containing a single point (because viewed as elements of $C(G^0)$, the $e_{ii}^{(k)}$ separate the points of G^0). Following the details in Example 3.25 of [PPW], we see that

$$e_{1,1}^{(1)} = e_{1,1}^{(2)} + e_{7,7}^{(2)} + e_{13,13}^{(2)} = e_{1,1}^{(3)} + e_{7,7}^{(3)} + e_{13,13}^{(3)} + e_{7,7}^{(2)} + e_{13,13}^{(2)} = \cdots$$

$$= \sum_{k=2}^{\infty} e_{7,7}^{(k)} + \sum_{k=2}^{\infty} e_{13,13}^{(k)} + \bigwedge_{k=1}^{\infty} e_{1,1}^{(k)} = \sum_{k=2}^{\infty} e_{7,7}^{(k)} + \sum_{k=2}^{\infty} e_{13,13}^{(k)}.$$

As $e_{7,8}^{(k)} \in \mathfrak{J}$ and $e_{13,14}^{(k)} \in \mathfrak{J}^*$ for all $k \geq 2$, $e_{11}^{(1)} \leq F_1 + F_2$. Hence $F_3 = 0$. Since g is a singleton, either $g \subseteq f_1$ or $g \subseteq f_2$. If $g \subseteq f_1$, then since $\{g_k \cap f_2\}_{k=1}^{\infty}$ is a decreasing sequence of compact sets with empty intersection, it follows that for some k, $k \geq 2$, $g_k \subseteq f_1$. But then $e_{12}^{(k)} \in \mathfrak{J}$, which is a contradiction. Similarly, the assumption that $g \subseteq f_2$ leads to a contradiction, so P is not closed. ∎

Our example of a maximal triangular subalgebra $\mathfrak{A}(P)$ of an AF C^*-algebra $C^*(G)$ which is not strongly maximal but has a closed P is complicated and will take awhile to develop. The key to the construction is

LEMMA 3.4. *Let $T \subseteq M(n)$ be a triangular subalgebra that is not maximal but is such that $T + T^*$ is a subalgebra of $M(n)$. Then there is a positive integer m, a subalgebra $S \subseteq M(m)$ and a unital imbedding $\sigma : M(n) \to M(m)$ satisfying the following conditions*:

(i) *S is a triangular subalgebra of $M(m)$ that is not maximal but is such that $S + S^*$ is an algebra.*

(ii) $\sigma(T) \subseteq S$.

(iii) *The D_m-bimodule generated by $S \cap \sigma(M(n))$ is equal to the D_m-bimodule generated by $\sigma(T)$ (where D_m is the algebra of diagonal matrices in $M(m)$).*

(iv) *For every triangular algebra $T \nsubseteq T_1 \subseteq M(n)$, the algebra generated by $\sigma(T_1)$ and S is not triangular.*

Befcre proving the lemma, we consider an example.

EXAMPLE 3.5. Let T be the diagonal matrices in $M(2)$ and let S be the subalgebra of $M(8)$ that is the linear span of the diagonal matrices and the matrix units $e_{2,3}$, $e_{4,1}$, $e_{5,8}$ and $e_{7,6}$. Let $\sigma : M(2) \to M(8)$ be the standard embedding. Conditions (i)–(iii) of Lemma 3.4 are immediate. As for (iv), note that there are only two possibilities: either T_1 is the full algebra of upper triangular matrices or the full algebra of lower triangular matrices. Suppose the first case holds. Then the D_8-bimodule generated by S and $\sigma(T_1)$ may be viewed as the subalgebra of the algebra of 4×4 matrices over $M(2)$ where the diagonal entires are upper triangular matrices, where the (2,1) and (1,2) blocks have zeros everywhere but in the lower left hand corners and where the (3,4) and (4,3) blocks have zeros everywhere but in the upper right hand corners. Write S_1 for the algebra generated by S and $\sigma(T_1)$. Then since e_{41} and e_{12} are in S_1, e_{42} and e_{23} are in S_1, e_{43} lies in S_1. But the bimodule generated by S and $\sigma(T_1)$ already contains e_{34}. Hence S_1 is not triangular. If, on the other other hand, T_1 were the algebra of lower triangular matrices, the lower 4×4 right hand corner of $M(8)$ could be used to show that S_1 is not triangular.

PROOF OF LEMMA 3.4. Note first that a subalgebra T of $M(n)$ containing D_n consists of all the matrices supported on a subset $Q \subseteq \{1,2,\dots,n\}^2$ satisfying $Q \circ Q \subseteq Q$ and $\Delta \subseteq Q$. That is, Q is a transitive and reflexive relation and

$$T = \{(a_{ij}) \in M(n) \mid a_{ij} = 0, \quad (i,j) \notin Q\}.$$

We call such a relation a *preorder*. The algebra T is triangular in $M(n)$ if and only if Q is antisymmetric, i.e., if and only if Q is a partial order. The algebra T is maximal triangular if and only if Q is a *total order*. Observe, too, that $T + T^*$ is an algebra (and T is triangular) if and only if the set $\{1,2,\ldots,n\}$ is a disjoint union of subsets, $\{1,2,\ldots,n\} = \cup_{k=1}^{l} A_k$, such that $Q \cap (A_k \times A_k)$ is a total order and $Q \cap (A_k \cap A_j) = \emptyset$, $k \neq j$.

Given $T \subseteq M(n)$ satisfying the hypotheses of Lemma 3.4, we write $A = \{1,2,\ldots,n\}$ and $A = \cup_{k=1}^{l} A_k$, where the A_k's partition A and $Q \cap (A_k \times A_k)$ is a total order for each k and $Q \cap (A_k \times A_j) = \emptyset$, $k \neq j$. (We will write $Q|A_k$ for $Q \cap (A_k \times A_k)$.) Let π_l be the group of permutations of l elements and let Ω be the set of all total orders on A that contains Q. For $p \in \pi_l$ and $a \in A$, we write $p(a)$ for the number j such that $a \in A_{p^{-1}(j)}$. The set Ω is finite and nonempty. Let N be its cardinality. Finally, set $m = n \times N \times l!$.

For every $p \in \pi_l$ and $\omega \in \Omega$, let $B(p,\omega) = A \times \{p\} \times \{\omega\}$ and let

$$B = \bigcup_{\substack{p \in \pi_l \\ \omega \in \Omega}} B(p,\omega) = A \times \pi_l \times \Omega.$$

Of course, the cardinality of B is m. Now define a partial order $F \subseteq B \times B$ by the prescription:

(i) For $p \in \pi_l$, $\omega \in \Omega$, $((a,p,\omega),(b,p,\omega)) \in F$ if and only if $(a,b) \in \omega$.

(ii) If $(p_1,\omega_1) \neq (p_2,\omega_2)$ in $(\pi_l \times \Omega)$, then $F \cap (B(p_1,\omega_1)) \times B(p_2,\omega_2)) = \emptyset$.

Thus F is the disjoint union of all the total orders on A extending Q, each counted with multiplicity $l!$. We write S for the corresponding subalgebra of $M(m)$. Then S is a triangular algebra that is not maximal, but $S + S^*$ is a subalgebra of $M(m)$.

Let τ be the permutation in π_l that is translation by 1 modulo l; i.e., $\tau(j) = j + 1 \pmod{l}$. We call two elements $p, q \in \pi_l$ equivalent and write $p \sim q$ if and only if $q = \tau^j p$ for some $j \in \mathbf{Z}$. Then $\sim$ is an equivalence relation. For $a, b \in A$, we define

$$\sigma(e_{a,b}) = \sum_{\substack{\omega \in \Omega \\ p \sim q,\, p(a) = q(b)}} e_{((a,p,\omega),(b,q,\omega))}.$$

This defines σ on matrix units and we want to show that σ extends by linearity to a unital $*$-homomorphism mapping $M(n)$ into $M(m)$. We note here that for every a and b in A and $p \in \pi_l$, there is a unique q in π_l such that $p \sim q$ and $p(a) = q(b)$. Thus the sum defining σ really only extends over $\Omega \times \pi_l$.

We have

$$\sigma(e_{ba}) = \sum_{\substack{\omega, p \sim q \\ p(b) = q(a)}} e_{((b,p,\omega),(a,q,\omega))} = \sum_{\substack{\omega, p \sim q \\ p(b) = q(a)}} e^*_{((a,q,\omega),(b,p,\omega))} = \sigma(e_{ab})^*.$$

Also,

$$\sigma(e_{ab})\sigma(e_{bc}) = \sum_{\substack{\omega,\omega' \\ p\sim q, p\sim q' \\ p(a)=q(b), p'(b)=q'(c)}} e_{((a,p,\omega),(b,q,\omega))} e_{((b,p'\omega'),(c,q',\omega'))}$$

$$= \sum_{\substack{\omega \\ p\sim q\sim q' \\ p(a)=q(b)=q'(c)}} e_{((a,p,\omega),(b,q,\omega))} e_{((b,q,\omega),(c,q',\omega))} = \sigma(e_{ac}).$$

Thus the linear extension of σ to $M(n)$ which we continue to denote by σ is a $*$-homomorphism from $M(n)$ into $M(m)$. To see that σ is unital, note that for $e_{aa} \in D_n$, $\sigma(e_{aa}) = \sum_{\omega,p} e_{((a,p,\omega),(a,p,\omega))}$ because when $p \sim q$ and $p(a) = q(a)$, necessarily $p = q$. Thus $\sigma(D_n) \subseteq D_m$ and

$$\sigma(I_n) = \sum_a \sigma(e_{aa}) = \sum_{a,\omega,p} e_{((a,p,\omega)),(a,p,\omega))} = I_m.$$

Next we prove that the D_m-bimodule generated by $S \cap \sigma(M(n))$ equals the D_m-bimodule generated by $\sigma(T)$, i.e., assertion (iii) of the lemma. For this, observe that the support of $\sigma(M_n)$ is the set

$$\{((a,p,\omega),(b,q,\omega)) \mid q \sim p, \quad q(b) = p(a), \quad \omega \in \Omega\}$$

and the support of S is the set

$$\{((a,p,\omega),(b,p,\omega)) \mid (a,b) \in \omega\}.$$

Their intersection, which is the support of the D_m-bimodule generated by $S \cap \sigma(M(n))$, is

$$\{((a,p,\omega),(b,p,\omega)) \mid (a,b) \in \omega, \quad p(a) = p(b)\}.$$

Since ω agrees with Q on each A_j, since $p(a) = p(b)$ if and only if a and b lie in the same A_j and since $Q \cap (A_j \times A_i) = \varnothing$, when $i \neq j$, this set is

$$\{((a,p,\omega),(b,p,\omega)) \mid (a,b) \in Q\}.$$

However, this set equals the support of $\sigma(T)$,

$$\{((a,p,\omega),(b,q,\omega)) \mid (a,b) \in Q, \quad q \sim p, \quad p(a) = q(b)\},$$

because if $(a,b) \in Q$, then $p(a) = p(b)$ and so the equation $p(a) = q(b)$ and the equivalence $q \sim p$ can happen simultaneously only if $p = q$. This proves (iii).

Since condition (iii) clearly implies (ii), we have only to verify (iv). Suppose that T_1 is a triangular subalgebra of $M(n)$ that contains T properly. Let Q_1 be its support. Then Q_1 is an order on A and $Q_1 \not\supseteq Q$. Pick an $(a,b) \in Q_1 \backslash Q$. Since Q induces a total order on each A_k, i.e., $Q|A_k$ is maximal, we have $a \in A_i$ and $b \in A_j$ with $i \neq j$. Select an $\omega_0 \in \Omega$ such that $(a,b) \in \omega_0^{-1}$. (It is easy to see that such an ω_0 exists.) We have $e_{ab} \in T_1$ by definition, so $\sigma(e_{ab})$ lies in $\sigma(T_1)$. But

$$\sigma(e_{ab}) = \sum_{\substack{\omega \in \Omega \\ p \sim q,\, p(a) = q(b)}} e_{((a,p,\omega),(b,q,\omega))},$$

so for every $p, q \in \pi_l$ with $p \sim q$ and $p(a) = q(b)$ we find that $e_{((a,p,\omega),(b,q,\omega))}$ lies in the D_m-bimodule generated by $\sigma(T_1)$ for all $\omega \in \Omega$. In particular, they all lie in the algebra S_1 generated by S and $\sigma(T_1)$. To see that S_1 is not triangular, recall that $\tau(j) = j + 1 (\mathrm{mod}\ l)$. So if $s = i - j$ (where $a \in A_i$ and $b \in A_j$), then $\tau^{ks} \sim \tau^{(k+1)s}$ for all $k \geq 0$ and $\tau^{ks}(a) = \tau^{ks}(i)$, by definition, which equals $i + k(i-j)(\mathrm{mod}\ l) = (k+1)(i-j) + j(\mathrm{mod}\ l) = \tau^{(k+1)s}(j)$, which equals $\tau^{(k+1)s}(b)$ by definition. Thus for all k, $e_{((a,\tau^{ks},\omega_0),(b,\tau^{(k+1)s},\omega_0))}$ lies in the D_m-bimodule generated by $\sigma(T_1)$. On the other hand, since $(b,a) \in \omega_0$ (because $(a,b) \in \omega_0^{-1}$), $e_{((b,\tau^{ks},\omega_0),(a,\tau^{ks},\omega_0))}$ belongs to S for all k. Hence

$$e_{((b,\tau^0,\omega_0),(a,\tau^{(l-1)s},\omega_0))} = e_{((b,\tau^0,\omega_0),(a,\tau^0,\omega_0))} e_{((a,\tau^0,\omega_0),(b,\tau^s,\omega_0))} e_{((b,\tau^s,\omega_0),(a,\tau^s,\omega_0))} \cdots e_{((b,\tau^{(l-1)s},\omega_0),(b,\tau^0,\omega_0))}$$

lies in the D_m-bimodule generated by $\sigma(T_1)$, and so S_1 is not triangular. ■

Given a D_n-bimodule R in $M(n)$ with support $Q \subseteq \{1,2,\ldots,n\}^2$, we call the D_n-bimodule whose support is $\{1,2,\ldots,n\}^2 \backslash Q$ the *complemented bimodule* of R. If T is a triangular subalgebra of $M(n)$, then we write $C(T)$ for the complemented bimodule of $T + T^*$.

Lemma 3.6. *With T, S, $M(n)$, $M(m)$ and σ as in Lemma 3.4, we have*
(i) $\sigma(T) = \sigma(M(n)) \cap S$,
(ii) $\sigma(T^*) = \sigma(M(n)) \cap S^*$, *and*
(iii) $\sigma(C(T)) = \sigma(M(n)) \cap C(S)$.

Proof. First note that if e_{ij} and e_{lp} are two different matrix units in $M(n)$, then the supports of $\sigma(e_{ij})$ and $\sigma(e_{lp})$ are disjoint. Indeed, suppose that (a,b) is the support of both $\sigma(e_{ij})$ and $\sigma(e_{lp})$. Then $e_{aa}\sigma(e_{ij})e_{bb} \neq 0$ and $e_{aa}\sigma(e_{lp})e_{bb} \neq 0$. Each of these products is a scalar multiple of e_{ab}, so, for some $c \in \mathbf{C}$, $e_{aa}\sigma(e_{ij} - ce_{lp})e_{bb} = 0$. But then

$$e_{aa}\sigma(e_{ij})e_{bb} = e_{aa}\sigma(e_{ii}(e_{ij} - ce_{lp})e_{jj})e_{bb} = \sigma(e_{ii})e_{aa}\sigma(e_{ij} - ce_{lp})e_{bb}\sigma(e_{jj}) = 0,$$

contrary to assumption.

By (ii) of Lemma 3.4, $\sigma(T) \subseteq \sigma(M(n)) \cap S$. Suppose e_{ij} is a matrix unit in $M(n)$ such that $\sigma(e_{ij}) \in \sigma(M(n)) \cap S$. Then the support of $\sigma(e_{ij})$ is contained in the support of $\sigma(M(n)) \cap S$ which, by (iii) of Lemma 3.4, is the support of $\sigma(T)$. Hence, by what we showed at the outset, $e_{ij} \in T$. Thus $\sigma(T) = \sigma(M(n)) \cap S$. Assertions (ii) and (iii) are now immediate. ■

EXAMPLE 3.7. There is an AF C^*-algebra, realized as $C^*(G)$ for an AF groupoid G, and a maximal triangular subalgebra of $C^*(G)$, realized as $\mathfrak{A}(P)$ for an open partial order $P \subseteq G$, such that P is also closed, but $P \cup P^{-1} \neq G$.

PROOF. We apply Lemma 3.4 inductively and use Lemma 3.6 to find a sequence

$$M(n_1) \xrightarrow{\alpha_1} M(n_2) \xrightarrow{\alpha_2} M(n_3) \xrightarrow{\alpha_3} \cdots$$

where each σ_i is a unital embedding with triangular subalgebras $T_i \subseteq M(n_i)$ satisfying the following conditions:

(i) For $i > 0$, T_i is not maximal triangular in $M(n_i)$, $\sigma_{i-1}(T_{i-1}) \subseteq T_i$ and $T_i + T_i^*$ is a subalgebra of $M(n_i)$.

(ii) $\sigma_i(T_i) = \sigma_i(M(n_i)) \cap T_{i+1}$, $\sigma_i(T_i^*) = \sigma_i(M(n_i)) \cap T_{i+1}^*$ and $\sigma_i(C(T_i)) = \sigma(M(n_i)) \cap C(T_{i+1})$.

(iii) The sequence $\{T_i\}$ is maximal in the sense that if $M(n_i) \supseteq S_i \supseteq T_i$, with each S_i a triangular subalgebra of $M(n_i)$ such that $\sigma_i(S_i) \subseteq S_{i+1}$, then $S_i = T_i$ for every i.

We write $C^*(G) = \varinjlim (M(n_i), \sigma_i)$ for a suitable AF groupoid G and apply the spectral theorem for bimodules, Theorem 3.10 of [MS], to write

$$\mathfrak{A}(P) = \varinjlim (T_i, \sigma_i)$$

$$\mathfrak{A}(P^{-1}) = \varinjlim (C(T_i^*), \sigma_i),$$

$$\mathfrak{A}(P_0) = \varinjlim (C(T_i), \sigma_i),$$

for open sets P and P_0. From Theorem 4.1 of [MS], we know P is a partial order. To complete the proof, we need to show that P is closed, $P \cup P^{-1} \neq G$ and $\mathfrak{A}(P)$ is maximal triangular.

To this end we use Proposition 2.6 of [PPW] and condition (ii) above to assert that $\mathfrak{A}(P) \cap M(n_i) = T_i$, $\mathfrak{A}(P^{-1}) \cap M(n_i) = T_i^*$ and $\mathfrak{A}(P_0) \cap M(n_i) = C(T_i)$. (Here, of course, we are viewing the $M(n_i)$, T_i and $C(T_i)$ as all contained in $C^*(G)$.) Write $P_1 = P \cup P^{-1} \cup P_0$, $P_2 = P \cap P_0$ and $P_3 = P^{-1} \cap P_0$. Then P_1, P_2

and P_3 are open subsets of G and since $C^*(G^0)$-bimodules are inductive, we have, for $j = 1,2,3$,

$$\mathfrak{A}(P_j) = \overline{\bigcup_i (\mathfrak{A}(P_j) \cap M(n_i))}.$$

Clearly $\mathfrak{A}(P_1) \supseteq \mathfrak{A}(P) + \mathfrak{A}(P^{-1}) + \mathfrak{A}(P_0)$, so $\mathfrak{A}(P_1) \cap M(n_i) \supseteq T_i + T_i^* + C(T_i) = M(n_i)$. Thus $\mathfrak{A}(P_1) = C^*(G)$ and $P \cup P^{-1} \cup P_0 = G$. Also, $\mathfrak{A}(P_2) \subseteq \mathfrak{A}(P) \cap \mathfrak{A}(P_0)$ and thus $\mathfrak{A}(P_2) \cap M(n_i) \subseteq T_i \cap C(T_i) = \{0\}$. Hence $P_2 = \varnothing$. Similarly, $P_3 = \varnothing$ and so $P_0 \cap (P \cup P^{-1}) = \varnothing$. Therefore, $P = G \setminus ((P^{-1} \setminus G^0) \cup P_0)$ is closed. Since $C(T_i) \neq 0$ for all i, $\mathfrak{A}(P_0) \neq \varnothing$. Thus $P_0 \neq \varnothing$ and $P \cup P^{-1} \nsubseteq G$.

To see that $\mathfrak{A}(P)$ is a maximal triangular subalgebra of $C^*(G)$, we first note that since each T_i is triangular in $M(n_i)$, $\mathfrak{A}(P)$ is triangular in $C^*(G)$. Suppose $\mathfrak{J}$ is a triangular subalgebra of $C^*(G)$ containing $\mathfrak{A}(P)$. Since $\mathfrak{J} \cap \mathfrak{J}^* = C^*(G^0)$, $\mathfrak{J}$ is a bimodule over $C^*(G^0)$. Thus by Lemma 1.3 of [P], $\mathfrak{J} = \cup(\mathfrak{J} \cap M(n_i))$. But for every $i > 0$, $M(n_i) \supseteq \mathfrak{J} \cap M(n_i) \supseteq \mathfrak{A}(P) \cap M(n_i) = T_i$ and $\mathfrak{J} \cap M(n_i)$ is triangular in $M(n_i)$ (because $(\mathfrak{J} \cap M(n_i)) \cap (\mathfrak{J} \cap M(n_i))^* = (\mathfrak{J} \cap \mathfrak{J}^*) \cap M(n_i) = D_{n_i}$). From condition (iii) above, $\mathfrak{J} \cap M(n_i) = T_i$ for all i, and so $\mathfrak{J} = \mathfrak{A}(P)$. ■

REMARK 3.8. We continue the notation of Example 3.7.

Let $\mathfrak{B} = \mathfrak{A}(P \cup P^{-1}) = \overline{\mathfrak{A}(P) + \mathfrak{A}(P)^*}$. Then $\mathfrak{B}$ is a C^*-subalgebra of $C^*(G)$. Indeed, $\mathfrak{B} = \overline{\cup_i(\mathfrak{A}(P \cup P^{-1}) \cap M(n_i))}$ and $\mathfrak{A}(P \cup P^{-1}) \cap M(n_i) = T_i + T_i^*$ is a C^*-subalgebra of $M(n_i)$. We claim that there is no closed subalgebra $\mathfrak{C}$ of $C^*(G)$, different from $\mathfrak{B}$, with diagonal equal to $\mathfrak{B}$, i.e., such that $\mathfrak{C} \cap \mathfrak{C}^* = \mathfrak{B}$. Let $\mathfrak{C}$ be a closed subalgebra of $C^*(G)$ with diagonal $\mathfrak{B}$ and write $\mathfrak{C} = \mathfrak{A}(Q)$ for an open set $Q \subseteq G$. We have $Q \cdot Q \subseteq Q$ and $Q \cap Q^{-1} = P \cup P^{-1}$. Set $R = (Q \setminus (P \cup P^{-1})) \cup P = Q \setminus (P^{-1} \setminus G^0)$. Then R is open, because P and P^{-1} are closed, and $R \cap R^{-1} = Q \cap Q^{-1} \setminus [(P^{-1} \cup P) \setminus G^0] = G^0$. If we can show that $R \cdot R \subseteq R$, then $\mathfrak{A}(R)$ will be a triangular subalgebra of $C^*(G)$ that contains $\mathfrak{A}(P)$. From the maximality of $\mathfrak{A}(P)$ it will follow that $P = R$ and hence that $Q = P \cup P^{-1}$.

To prove that $R \cdot R \subseteq R$, set $L = Q \setminus (P \cup P^{-1})$, so $R = L \cup P$. It suffices to show that $L \cdot L \subseteq R$, $L \cdot P \subseteq R$ and $P \cdot L \subseteq R$. Suppose x and y are composable elements of L. If $xy \notin R$, then $xy \in P^{-1} \setminus G^0$, since $xy \in Q \cdot Q \subseteq Q$. But then, $y = x^{-1}xy$ lies in $L^{-1} \cdot P^{-1} \subseteq Q^{-1} \cdot Q^{-1} \subseteq Q^{-1}$. Hence y lies in

$$Q^{-1} \cap L = Q^{-1} \cap (Q \setminus (P \cup P^{-1})) = Q^{-1} \cap Q \setminus (P \cap P^{-1}) = \varnothing.$$

Hence $L \cdot L \subseteq R$. If, now, $x \in L$ and $y \in P$ are composable elements and if $xy \notin R$, then $xy \in P^{-1} \setminus G^0$. Therefore, $x = xyy^{-1}$ lies in $P^{-1} \cdot P^{-1} \subseteq P^{-1}$. But $L \cap P^{-1} = \varnothing$, so $L \cdot P \subseteq R$. The argument showing $P \cdot L \subseteq R$ is similar and so we omit it, completing the proof. ■

REFERENCES

[A] Wm. B. Arveson, *Analyticity in operator algebras*, Am. J. Math. **89** (1967), 578-642.

[KT] S. Kawamura and J. Tomiyama, *On subdiagonal algebras associated with flows on operator algebras*, J. Math. Soc. Japan **29** (1977), 73-90.

[MS] P. S. Muhly and Baruch Solel, *Subalgebras of groupoid C^*-algebras*, J. Reine Angew. Math. **402** (1989), 41-75.

[PPW] J. Peters, Y. Poon and B. Wagner, *Triangular AF algebras*, J. Operator Theory, to appear.

[P] S. C. Power, *On ideals of nest subalgebras of C^*-algebras*, Proc. London Math. Soc. (2) **50** (1985), 314-332.

[R] J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Math. **793**, Springer-Verlag, Berlin-Heidelberg-New York, 1980.

[T] M. A. Thelwall, *Maximal triangular subalgebras of AF algebras*, preprint.

[V] B. A. Ventura, *A note on subdiagonality for triangular AF algebras*, preprint.